JOURNAL OF ALGEBRA 15, 589–595 (1970)

# Orthogonal Groups over Global Fields of Characteristic 2*

BARTH POLLAK

*University of Notre Dame, Notre Dame, Indiana 46556*

*Communicated by J. Dieudonné*

Received October 1, 1969

The relationship between the spinorial kernel and the commutator subgroup of the orthogonal group is of basic importance in the study of the structure of the orthogonal group and has been worked out for local and global fields of characteristic $\neq 2$ (see [5] and [7]). In [3], E. Connors treats the case of local fields of characteristic 2. In this paper we deal with global fields of characteristic 2.

Let $V$ be a nondefective quadratic space over such a field. Let $O(V)$ denote the orthogonal group of $V$, $O'(V)$ its spinorial kernel, and $\Omega(V)$ its commutator subgroup. We prove the following

THEOREM. $O'(V) = \Omega(V)$.

Since $V$ is isotropic if $\dim V > 4$, the interesting case is when $V$ is 4-dimensional anisotropic. The problem comes down to the construction of a 2-dimensional nondefective space which represents certain numbers and is itself represented by $V$. To accomplish this, we need to use the Very Strong Approximation theorem and the Reciprocity law. This is in contrast to the case of characteristic $\neq 2$ where it is enough to construct a certain 1-dimensional space which is represented by $V$, a task for which the Weak Approximation theorem alone suffices.

## 1. PRELIMINARIES

Throughout this paper, $F$ will denote a field of characteristic 2 and $\wp$ the additive homomorphism of $F$ into itself given by $\wp(x) = x^2 + x$. A *quadratic space* $V$ over $F$ is a finite dimensional vector space over $F$ together with a

* Research partially supported by National Science Foundation Grant GP-11342.

quadratic form $Q$ and associated symmetric bilinear form $B$. We assume the reader is familiar with the theory of such spaces (see [1], [4], [9]).

$$K(V) = \{x \in V \mid B(x, V) = 0\}$$

is called the *core* of $V$ and $V$ is called *nondefective* if $K(V) = 0$.

$$R(V) = \{x \in K(V) \mid Q(x) = 0\}$$

is called the *radical* of $V$ and $V$ is called *nondegenerate* if $R(V) = 0$. We shall always assume that the given quadratic spaces are nondegenerate unless we specifically state otherwise.

The 1-dimensional quadratic space $Fx$ with $Q(x) = a$ will sometimes be denoted $(a)$. Similarly, if $V$ is a 2-dimensional quadratic space with basis $x$, $y$ such that $Q(x) = a$, $Q(y) = c$ and $B(x, y) = b \neq 0$ we sometimes write $V = ({}_a{}^b{}_c)$. A space $V = ({}_0{}^1{}_0)$ is called a *hyperbolic plane*. An orthogonal sum of hyperbolic planes is called a *hyperbolic space*. A nonzero $x \in V$ with $Q(x) = 0$ is called *isotropic* and a space which contains an isotropic vector is called *isotropic*.

Suppose $V$ is a nondefective space. Then $V$ has an orthogonal splitting

$$V = \begin{pmatrix} & b_1 & \\ a_1 & & c_1 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} & b_n & \\ a_n & & c_n \end{pmatrix}.$$

The coset $\sum_i a_i c_i b_i^{-2} + \wp(F)$ is an invariant of $V$, the *Arf invariant* (see [1]). We shall denote it $\Delta V$. We will sometimes find it convenient to write $\Delta V = a$, $a \in F$, instead of $\Delta V = a + \wp(F)$.

Let $U$ and $V$ be quadratic spaces over $F$. A *representation* of $U$ into $V$ is a linear map $\sigma : U \to V$ such that $Q(\sigma x) = Q(x)$ for all $x \in U$. Usually we supress $\sigma$ and simply write $U \to V$. A representation is always injective (remember that our spaces are assumed to be nondegenerate). If the representation is bijective we call it an *isometry*, say that $U$ and $V$ are *isometric* and write $U \cong V$. If $U = V$, the set of all isometries of $V$ into itself form a group, the *orthogonal group* of $V$ denoted $O(V)$. With a single exception (when dim $V = 4$ and $F$ is the finite field of 2 elements), $O(V)$ is generated by orthogonal transvections (see [4]).

## 2. Local Results

In this section we assume that $F$ is a local field. Thus $F$ is complete with respect to a discrete valuation with finite residue class field $f$. In this situation, $f/\wp f$ has two elements. We let 0 and $l$ denote representatives in $F$ of these elements. Also let $\mathfrak{o}$ denote the valuation ring of $F$ and $\mathfrak{u}$ the set of units of $F$.

LEMMA 2.1. *Let $a, b \in \mathfrak{o}$. Then*

$$\begin{pmatrix} & 1 \\ a & & b \end{pmatrix} \cong \begin{pmatrix} & 1 \\ 0 & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & 1 \\ 1 & & l \end{pmatrix}.$$

*Proof.* Set $P = ({}_a{}^1{}_b)$, $H = ({}_0{}^1{}_0)$ and $L = ({}_1{}^1{}_l)$. Now $\Delta(P) = ab + \wp(F)$. If either $a$ or $b \notin \mathfrak{u}$ then, by Lemma 3.2 in Chapter IV of [9], $\Delta(P) = 0$ and then $P \cong H$ by Zusatz 2 of [1]. If both $a, b \in \mathfrak{u}$, then either $\Delta(P) = 0$ in which case $P \cong H$ as before or else $\Delta(P) = l + \wp(F)$ by Lemma 3.2 in Chapter IV of [9]. By Lemma 1.6 of [8] $Q(P) = Q(L)$. Hence $P \cong L$ by Theorem 5.4 in Chapter II of [9].

PROPOSITION 2.2. *Suppose $P = ({}_a{}^1{}_b)$ with $a, b \in \mathfrak{o}$ and $V = P_1 \perp P_2$ with $P_i = ({}_{a_i}{}^1{}_{b_i})$, $a_i, b_i \in \mathfrak{u}$ for $i = 1, 2$. Then $P \to V$.*

*Proof.* Immediate by Lemma 2.1 and Lemma 4.6 in Chapter II of [9].
For the convenience of the reader we record four results.

PROPOSITION 2.3. *Let $U$ and $V$ be nondefective quadratic spaces over $F$ with* $\dim V - \dim U = 2$. *Then $\Delta U \neq \Delta V$ implies $U \to V$.*

*Proof.* See Theorem 3.6 in Chapter IV of [9].

PROPOSITION 2.4. *Let $V$ be a quaternary anisotropic quadratic space over $F$. Then $\Delta V = 0$.*

*Proof.* See Corollary 3.7 in Chapter IV of [9].

PROPOSITION 2.5. *The anisotropic plane $({}_1{}^1{}_l)$ represents all elements of $F$ of even order.*

*Proof.* See Lemma 1.6 of [8].

PROPOSITION 2.6. *Let $b_1, b_2 \in F$ and suppose that exactly one of $b_1, b_2, b_1b_2$ has even order. Then there exists $d \in F$ such that $\{1, b_1, b_2, b_1b_2\} \subseteq Q(P)$ where $P = ({}_1{}^1{}_d)$.*

*Proof.* See [3].
We shall refer to the plane of Proposition 2.6 as a *Connors plane* for $b_1, b_2$ in Section 4.

## 3. GLOBAL RESULTS

In this section we examine the Hasse principle for representation of quadratic spaces over global fields. Although the result is well known in the characteristic $\neq 2$ case, I have not found an explicit statement in the literature let alone proofs in the characteristic 2 case.

So let $F$ be a global field. If $\mathfrak{p}$ is a prime of $F$ we denote, as usual, the completion of $F$ at $\mathfrak{p}$ by $F_{\mathfrak{p}}$ and the localization of $V$ at $\mathfrak{p}$ by $V_{\mathfrak{p}}$.

We first of all note the following

PROPOSITION 3.1. *Let $V$ be a quadratic space over $F$. If* $\dim V > 4$, *then $V$ is isotropic.*

*Proof.* Immediate by Lemma 4.5 in Chapter II of [9] and the bottom of p. 58 of [6].

THEOREM 3.2. (*Hasse Principle for isotropy*). *Let $V$ be a quadratic space* (*possibly degenerate*) *over $F$. Then $V$ is isotropic if and only if $V_{\mathfrak{p}}$ is isotropic for all primes $\mathfrak{p}$ of $F$.*

*Proof.* This result is implicit in the literature so we merely sketch a proof. First of all, the theorem is trivial if $V$ is degenerate. So we may assume that $V$ is nondegenerate. Since nondegenerate spaces over global fields are automatically isotropic if $\dim V > 4$ or if $\dim V = 4$ and $V$ is defective, we may assume that $\dim V \leqslant 4$ and that $V$ is nondefective if $\dim V = 4$. If $\dim V = 1$ the result is trivial. If $\dim V = 2$, then the theorem is an immediate consequence of Theorem 1 in Chapter 9 of [2] when $V$ is defective and Theorem 2 in Chapter 5 of [2] when $V$ is nondefective. For $\dim V = 3$ or 4 the theorem follows readily from Satz 9 and Satz 10 of [1], the Brauer–Hasse–Noether theorem, and the following easily verified facts: $K(V)_{\mathfrak{p}} = K(V_{\mathfrak{p}})$ and $R(V) = 0$ implies $R(V_{\mathfrak{p}}) = 0$.

An immediate consequence is

COROLLARY 3.3. *Let $a \neq 0$ be in $F$ and let $V$ be any quadratic space over $F$. Then $(a) \to V$ if and only if $(a)_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.*

*Proof.* Use Lemma 4.4 in Chapter II of [9] and Theorem 3.2.

Let $H(k)$ denote the hyperbolic space of dimension $2k$. We have

LEMMA 3.4. *Let $V$ be a quadratic space over $F$. Then $H(k) \to V$ if and only if $H(k)_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.*

*Proof.* $H(k) \to V$ implies $H(k)_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all $\mathfrak{p}$ is trivial. The converse follows readily by induction on $k$ utilizing Theorem 3.2 and Witt's cancellation theorem.

LEMMA 3.5. *Let $P$ be a nondefective plane. Then, for any quadratic space $V$, $P \to V$ if and only if $H(2) \to V \perp P$.*

*Proof.* $P \to V$ if and only if $V \cong P \perp W$ for some space $W$. Now $V \perp P \cong P \perp P \perp W \cong H(2) \perp W$ by Lemma 4.6 in Chapter II of [9] and this is true if and only if $H(2) \to V \perp P$.

LEMMA 3.6. *Let $P$ be a nondefective plane. Then, for any quadratic space $V$, $P \to V$ if and only if $P_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.*

*Proof.* Immediate by Lemmas 3.4 and 3.5.

We now have the Hasse principle for representation of nondefective spaces.

PROPOSITION 3.7. *Let $U$ be a nondefective space and $V$ any quadratic space. Then $U \to V$ if and only if $U_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.*

*Proof.* $U \to V$ implies $U_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for all $\mathfrak{p}$ is trivial. We shall establish the converse. Now $U$ nondefective implies $U = P_1 \perp \cdots \perp P_k$ where each $P_i$ is a nondefective plane. The proof follows easily by induction on $k$ utilizing Lemma 3.6 and Witt's cancellation theorem.

## 4. THE MAIN RESULT

In this section, $F$ will signify a global field and $V$ a nondefective space over $F$. We denote by $F_{\mathfrak{p}}$ the completion of $F$ at the prime $\mathfrak{p}$, by $\mathfrak{o}_{\mathfrak{p}}$ its valuation ring, by $\mathfrak{u}_{\mathfrak{p}}$ the set of units of $F_{\mathfrak{p}}$ and by $l_{\mathfrak{p}}$ the $l$ of $F_{\mathfrak{p}}$ as defined in Section 2. We use the letter $Q$ to stand for the quadratic form on any quadratic space in this section. $O(V)$, as usual, will denote the orthogonal group of $V$, $O'(V)$ its spinorial kernel and $\Omega(V)$ its commutator subgroup. Our goal is to prove the following

THEOREM 4.1. $O'(V) = \Omega(V)$.

*Proof.* The standard argument works if $\dim V < 4$ or $V$ is isotropic. Since $F$ is a global field, $V$ is isotropic if $\dim V > 4$. Hence we may assume that $\dim V = 4$ and $V$ is anisotropic.

We may write

$$V = \begin{pmatrix} & 1 \\ a_1 & a_2 \end{pmatrix} \perp \begin{pmatrix} & 1 \\ a_3 & a_4 \end{pmatrix}.$$

Now let $\sigma \in O'(V)$. We must show that $\sigma \in \Omega(V)$. We may express $\sigma$ as the product of either 2 or 4 orthogonal transvections (see [4], 1) of Section 10). If 2, then clearly $\sigma \in \Omega(V)$ so we may assume that $\sigma$ is the product of 4 orthogonal transvections,

$$\sigma = \tau_{x_1}\tau_{x_2}\tau_{x_3}\tau_{x_4}$$

and without loss of generality write $Q(x_1) = 1$, $Q(x_2) = b_1$, $Q(x_3) = b_2$, $Q(x_4) = b_1 b_2$. The proof will be accomplished by constructing a nondefective plane $P$ such that

(a) $P \to V$

(b) $\{1, b_1, b_2, b_1 b_2\} \subseteq Q(P)$

for then, by a standard argument, $\sigma$ will be conjugate to an element of $\Omega(V)$ and hence be in $\Omega(V)$ itself.

We pick a finite set of primes $S$ of $F$ as follows: $\mathfrak{p} \in S$ if and only if

(1) $V_\mathfrak{p}$ is anisotropic and all three of $b_1$, $b_2$, $b_1 b_2$ have even order at $\mathfrak{p}$.

(2) $V_\mathfrak{p}$ is anisotropic and exactly one of $b_1$, $b_2$, $b_1 b_2$ has even order at $\mathfrak{p}$. (Note that when $V_\mathfrak{p}$ is anisotropic either (1) or (2) obtains.)

(3) $V_\mathfrak{p}$ is isotropic and at least one of the $a_i$ is a nonunit at $\mathfrak{p}$ or at least one of $b_1$, $b_2$, $b_1 b_2$ has odd order at $\mathfrak{p}$.

Now, for each $\mathfrak{p} \in S$, we select $d_\mathfrak{p} \in F_\mathfrak{p}$ as follows:

(1) Select for $d_\mathfrak{p}$ the $l$ of Proposition 2.5.

(2) Select $d_\mathfrak{p}$ so that $\begin{pmatrix} 1 & 1 \\ 1 & d_\mathfrak{p} \end{pmatrix}$ is a Connors plane for $b_1$, $b_2$.

(3) Pick any $d_\mathfrak{p} \in \wp(F_\mathfrak{p})$.

Finally let $\mathfrak{q} \notin S$. Then, since $\wp(F_\mathfrak{p})$ is open in $F_\mathfrak{p}$, by the Very Strong Approximation theorem (see [6], 33.11) there exists $d \in F$ such that

(i) $d \equiv d_\mathfrak{p} \bmod \wp(F_\mathfrak{p})$ for all $\mathfrak{p} \in S$

(ii) $d \in \mathfrak{o}_\mathfrak{p}$ for all $\mathfrak{p} \notin S \cup \mathfrak{q}$.

Set $P = \begin{pmatrix} 1 & 1 \\ 1 & d \end{pmatrix}$. We now show that conditions (a) and (b) are satisfied.

Verification of (a): By Proposition 3.7 it is enough to show that $P_\mathfrak{p} \to V_\mathfrak{p}$ for all $\mathfrak{p}$.

(1) Suppose $V_\mathfrak{p}$ is anisotropic. Then $\mathfrak{p} \in S$ of type (1) or (2). But then $P_\mathfrak{p}$ is anisotropic by construction. Hence $\Delta P_\mathfrak{p} \neq 0$ by Zusatz 2 of [1] and $\Delta V_\mathfrak{p} = 0$ by Proposition 2.4. Hence $P_\mathfrak{p} \to V_\mathfrak{p}$ by Proposition 2.3.

(2) Suppose $V_\mathfrak{p}$ is isotropic and $\mathfrak{p} \in S$. Then $P_\mathfrak{p}$ is a hyperbolic plane by construction. But then $P_\mathfrak{p} \to V_\mathfrak{p}$ by Satz 6 of [1].

(3) Suppose $V_\mathfrak{p}$ is isotropic and $\mathfrak{p} \notin S \cup \mathfrak{q}$. Invoke Proposition 2.2.

(4) Finally let $\mathfrak{p} = \mathfrak{q}$. Then $V_\mathfrak{q}$ is isotropic. We also note that all $a_i \in \mathfrak{u}_\mathfrak{q}$. If $d \in \mathfrak{o}_\mathfrak{q}$, then $P_\mathfrak{q} \to V_\mathfrak{q}$ by Proposition 2.2. If $d \notin \mathfrak{o}_\mathfrak{q}$, then $\Delta P_\mathfrak{q} \neq \Delta V_\mathfrak{q}$ by Lemma 3.2 in Chapter IV of [9] and hence $P_\mathfrak{q} \to V_\mathfrak{q}$ by Proposition 2.3.

Thus $P_\mathfrak{p} \to V_\mathfrak{p}$ for all $\mathfrak{p}$ and (a) is established.

Verification of (b): By Corollary 3.3 it is enough to show that $\{1, b_1, b_2, b_1b_2\} \subseteq Q(P_\mathfrak{p})$ for all $\mathfrak{p}$. We consider five cases:

(1) $\mathfrak{p} \in S$ of type 1. By Proposition 2.5 and our choice of $d$.

(2) $\mathfrak{p} \in S$ of type 2. By construction $P_\mathfrak{p}$ is a Connors plane for $b_1$, $b_2$.

(3) $\mathfrak{p} \in S$ of type 3. In this case $P_\mathfrak{p}$ is a hyperbolic plane by construction and hence universal.

(4) $\mathfrak{p} \notin S \cup \mathfrak{q}$. Then, since $d \in \mathfrak{o}_\mathfrak{p}$, $P_\mathfrak{p}$ is either a hyperbolic plane and hence universal or else $d \equiv l_\mathfrak{p} \bmod \wp(F_\mathfrak{p})$ by Lemma 2.1 and hence represents $1, b_1, b_2, b_1b_2$ by Proposition 2.5 (since $1, b_1, b_2, b_1b_2$ are units in $F_\mathfrak{p}$).

(5) Finally suppose $\mathfrak{p} = \mathfrak{q}$. Note that $Q(P)$ is the norm group of a separable quadratic extension of $F$ (since $P$ is nondefective) and $\{1, b_1, b_2, b_1b_2\} \subseteq Q(P_\mathfrak{p})$ for all $\mathfrak{p} \neq \mathfrak{q}$. Then $\{1, b_1, b_2, b_1b_2\} \subseteq Q(P_\mathfrak{q})$ by the reciprocity law of global class field theory. Q.E.D.

## References


1. C. Arf, Untersuchungen über quadratische Formen in Körpern der Charakteristik 2, *J. Reine Angew. Math.* **183** (1940), 148–167.
2. E. Artin and J. Tate, "Class Field Theory," Benjamin, New York, 1968.
3. E. Connors, The structure of $O'/\Omega$ over local fields of characteristic 2, *Proc. Amer. Math. Soc.*, **22** (1969), 596–599.
4. J. Dieudonné, "La Géométrie des Groupes Classiques," Second Édition, Springer–Verlag, Berlin/New York, 1963.
5. M. Kneser, Orthogonale Gruppen über Algebraischen Zahlkörpern, *J. Reine Angew. Math.* **196** (1956), 213–220.
6. O. T. O'Meara, "Introduction to Quadratic Forms," Springer–Verlag, Berlin/New York, 1963.
7. B. Pollak, 4-dimensional orthogonal groups over algebraic number fields. *J. Reine Angew. Math.* **211** (1962), 176–178.
8. C. R. Riehm, Integral representations of quadratic forms in characteristic 2, *Amer. J. Math.* **87** (1965), 32–64.
9. C. H. Sah, Integral equivalence of quadratic forms in local fields of characteristic 2, Ph.D. Thesis, Princeton University, Princeton, N. J., 1959.